
Introduction to the Laplace Transform

Assessment Problems

AP 12.1 [a] $\cosh \beta t = \frac{e^{\beta t} + e^{-\beta t}}{2}$

Therefore,

$$\begin{aligned} \mathcal{L}\{\cosh \beta t\} &= \frac{1}{2} \int_{0^-}^{\infty} [e^{(s-\beta)t} + e^{-(s+\beta)t}] dt \\ &= \frac{1}{2} \left[\frac{e^{-(s-\beta)t}}{-(s-\beta)} \Big|_{0^-}^{\infty} + \frac{e^{-(s+\beta)t}}{-(s+\beta)} \Big|_{0^-}^{\infty} \right] \\ &= \frac{1}{2} \left(\frac{1}{s-\beta} + \frac{1}{s+\beta} \right) = \frac{s}{s^2 - \beta^2} \end{aligned}$$

[b] $\sinh \beta t = \frac{e^{\beta t} - e^{-\beta t}}{2}$

Therefore,

$$\begin{aligned} \mathcal{L}\{\sinh \beta t\} &= \frac{1}{2} \int_{0^-}^{\infty} [e^{-(s-\beta)t} - e^{-(s+\beta)t}] dt \\ &= \frac{1}{2} \left[\frac{e^{-(s-\beta)t}}{-(s-\beta)} \Big|_{0^-}^{\infty} - \frac{1}{2} \left[\frac{e^{-(s+\beta)t}}{-(s+\beta)} \Big|_{0^-}^{\infty} \right] \right] \\ &= \frac{1}{2} \left(\frac{1}{s-\beta} - \frac{1}{s+\beta} \right) = \frac{\beta}{(s^2 - \beta^2)} \end{aligned}$$

AP 12.2 [a] Let $f(t) = te^{-at}$:

$$F(s) = \mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

Now, $\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$

$$\text{So, } \mathcal{L}\{t \cdot te^{-at}\} = -\frac{d}{ds} \left[\frac{1}{(s+a)^2} \right] = \frac{2}{(s+a)^3}$$

[b] Let $f(t) = e^{-at} \sinh \beta t$, then

$$\mathcal{L}\{f(t)\} = F(s) = \frac{\beta}{(s+a)^2 - \beta^2}$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-) = \frac{s(\beta)}{(s+a)^2 - \beta^2} - 0 = \frac{\beta s}{(s+a)^2 - \beta^2}$$

[c] Let $f(t) = \cos \omega t$. Then

$$F(s) = \frac{s}{(s^2 + \omega^2)} \quad \text{and} \quad \frac{dF(s)}{ds} = \frac{-(s^2 - \omega^2)}{(s^2 + \omega^2)^2}$$

$$\text{Therefore } \mathcal{L}\{t \cos \omega t\} = -\frac{dF(s)}{ds} = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

AP 12.3

$$F(s) = \frac{6s^2 + 26s + 26}{(s+1)(s+2)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = \frac{6 - 26 + 26}{(1)(2)} = 3; \quad K_2 = \frac{24 - 52 + 26}{(-1)(1)} = 2$$

$$K_3 = \frac{54 - 78 + 26}{(-2)(-1)} = 1$$

$$\text{Therefore } f(t) = [3e^{-t} + 2e^{-2t} + e^{-3t}]u(t)$$

AP 12.4

$$F(s) = \frac{7s^2 + 63s + 134}{(s+3)(s+4)(s+5)} = \frac{K_1}{s+3} + \frac{K_2}{s+4} + \frac{K_3}{s+5}$$

$$K_1 = \frac{63 - 189 - 134}{1(2)} = 4; \quad K_2 = \frac{112 - 252 + 134}{(-1)(1)} = 6$$

$$K_3 = \frac{175 - 315 + 134}{(-2)(-1)} = -3$$

$$f(t) = [4e^{-3t} + 6e^{-4t} - 3e^{-5t}]u(t)$$

AP 12.5

$$F(s) = \frac{10(s^2 + 119)}{(s + 5)(s^2 + 10s + 169)}$$

$$s_{1,2} = -5 \pm \sqrt{25 - 169} = -5 \pm j12$$

$$F(s) = \frac{K_1}{s + 5} + \frac{K_2}{s + 5 - j12} + \frac{K_2^*}{s + 5 + j12}$$

$$K_1 = \frac{10(25 + 119)}{25 - 50 + 169} = 10$$

$$K_2 = \frac{10[(-5 + j12)^2 + 119]}{(j12)(j24)} = j4.17 = 4.17/90^\circ$$

Therefore

$$\begin{aligned} f(t) &= [10e^{-5t} + 8.33e^{-5t} \cos(12t + 90^\circ)] u(t) \\ &= [10e^{-5t} - 8.33e^{-5t} \sin 12t] u(t) \end{aligned}$$

AP 12.6

$$F(s) = \frac{4s^2 + 7s + 1}{s(s + 1)^2} = \frac{K_0}{s} + \frac{K_1}{(s + 1)^2} + \frac{K_2}{s + 1}$$

$$K_0 = \frac{1}{(1)^2} = 1; \quad K_1 = \frac{4 - 7 + 1}{-1} = 2$$

$$\begin{aligned} K_2 &= \frac{d}{ds} \left[\frac{4s^2 + 7s + 1}{s} \right]_{s=-1} = \frac{s(8s + 7) - (4s^2 + 7s + 1)}{s^2} \Big|_{s=-1} \\ &= \frac{1 + 2}{1} = 3 \end{aligned}$$

$$\text{Therefore } f(t) = [1 + 2te^{-t} + 3e^{-t}] u(t)$$

AP 12.7

$$\begin{aligned} F(s) &= \frac{40}{(s^2 + 4s + 5)^2} = \frac{40}{(s + 2 - j1)^2(s + 2 + j1)^2} \\ &= \frac{K_1}{(s + 2 - j1)^2} + \frac{K_2}{(s + 2 - j1)} + \frac{K_1^*}{(s + 2 + j1)^2} \\ &\quad + \frac{K_2^*}{(s + 2 + j1)} \end{aligned}$$

$$K_1 = \frac{40}{(j2)^2} = -10 = 10/180^\circ \quad \text{and} \quad K_1^* = -10$$

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \left[\frac{s^3[4 + (7/s) + (1/s)^2]}{s^3[1 + (1/s)]^2} \right] = 4$$

$$\therefore f(0^+) = 4$$

$$\lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} \left[\frac{4s^2 + 7s + 1}{(s + 1)^2} \right] = 1$$

$$\therefore f(\infty) = 1$$

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \left[\frac{40s}{s^4[1 + (4/s) + (5/s^2)]^2} \right] = 0$$

$$\therefore f(0^+) = 0$$

$$\lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} \left[\frac{40s}{(s^2 + 4s + 5)^2} \right] = 0$$

$$\therefore f(\infty) = 0$$

Problems

P 12.1 [a] $f(t) = 120 + 30t \quad -4\text{ s} \leq t \leq 0$

$$f(t) = 120 - 30t \quad 0 \leq t \leq 8\text{ s}$$

$$f(t) = -360 + 30t \quad 8\text{ s} \leq t \leq 12\text{ s}$$

$$f(t) = 0 \quad \text{elsewhere}$$

$$f(t) = (120 + 30t)[u(t + 4) - u(t)] + (120 - 30t)[u(t) - u(t - 8)] \\ + (-360 + 30t)[u(t - 8) - u(t - 12)]$$

[b] $f(t) = 50 \sin \frac{\pi}{2}t [u(t) - u(t - 4)]$

$$= (50 \sin \frac{\pi}{2}t)u(t) - (50 \sin \frac{\pi}{2}t)u(t - 4)$$

[c] $f(t) = (30 - 3t)t[u(t) - u(t - 10)]$

P 12.2 [a] $(50 + 2.5t)[u(t + 20) - u(t)] + (50 - 5t)[u(t) - u(t - 10)]$

$$= (2.5t + 50)u(t + 20) - 2.5tu(t) + (5t - 50)u(t - 10)$$

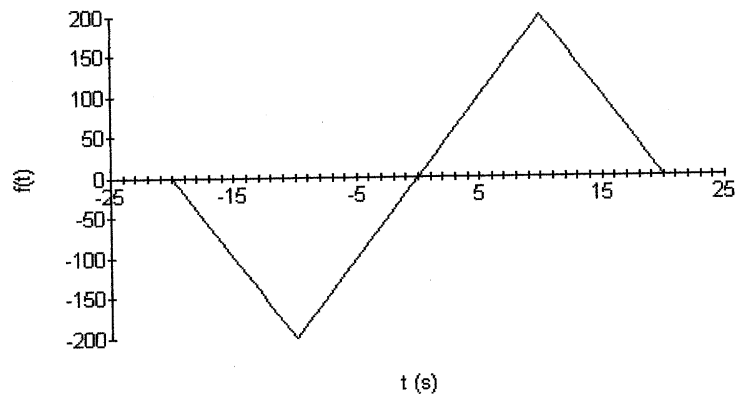
[b] $(5t + 45)[u(t + 9) - u(t + 6)] + 15[u(t + 6) - u(t + 3)] - 5t[u(t + 3) - u(t - 3)]$

$$- 15[u(t - 3) - u(t - 6)] + (5t - 45)[u(t - 6) - u(t - 9)]$$

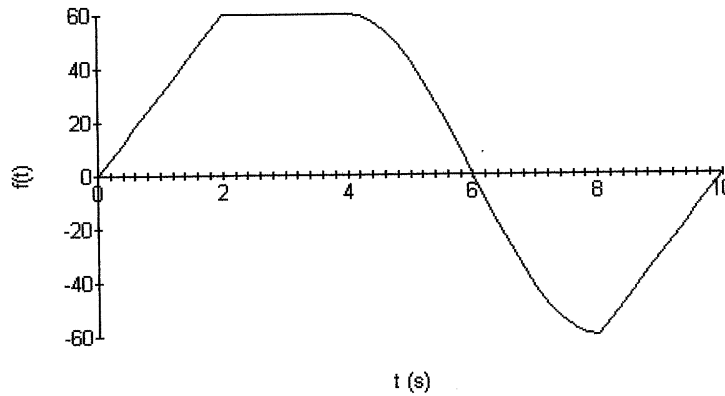
$$= 5(t + 9)u(t + 9) - 5(t + 6)u(t + 6) - 5(t + 3)u(t + 3) + 5(t - 3)u(t - 3)$$

$$+ 5(t - 6)u(t - 6) - 5(t - 9)u(t - 9)$$

P 12.3



P 12.4 [a]



$$\begin{aligned}
 \text{[b]} \quad f(t) &= 30t[u(t) - u(t-2)] + 60[u(t-2) - u(t-4)] \\
 &\quad + 60 \cos\left(\frac{\pi}{4}t - \pi\right)[u(t-4) - u(t-8)] \\
 &\quad + (30t - 300)[u(t-8) - u(t-10)]
 \end{aligned}$$

$$\text{P 12.5 [a]} \quad A = \left(\frac{1}{2}\right)bh = \left(\frac{1}{2}\right)(2\varepsilon)\left(\frac{1}{\varepsilon}\right) = 1.0$$

$$\text{[b]} \quad 0; \quad \text{[c]} \quad \infty$$

$$\begin{aligned}
 \text{P 12.6 [a]} \quad I &= \int_{-2}^4 (t^3 + 4)\delta(t) dt + \int_{-2}^4 4(t^3 + 4)\delta(t-2) dt \\
 &= 4 + 4(8 + 4) = 52
 \end{aligned}$$

$$\begin{aligned}
 \text{[b]} \quad I &= \int_{-3}^4 t^2\delta(t) dt + \int_{-3}^4 t^2\delta(t+2.5) dt + 0 \\
 &= 0^2 + (-2.5)^2 + 0 = 6.25
 \end{aligned}$$

$$\text{P 12.7} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(3+j\omega)}{(4+j\omega)} \cdot \pi\delta(\omega) \cdot e^{jt\omega} d\omega = \left(\frac{1}{2\pi}\right) \left(\frac{3+j0}{4+j0}\pi e^{-j0t}\right) = \frac{3}{8}$$

P 12.8 As $\varepsilon \rightarrow 0$ the amplitude $\rightarrow \infty$; the duration $\rightarrow 0$; and the area is independent of ε , i.e.,

$$A = \int_{-\infty}^{\infty} \frac{\varepsilon}{\pi \varepsilon^2 + t^2} dt = 1$$

$$\text{P 12.9} \quad F(s) = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} e^{-st} dt = \frac{e^{s\varepsilon} - e^{-s\varepsilon}}{2\varepsilon s}$$

$$F(s) = \frac{1}{2s} \lim_{\varepsilon \rightarrow 0} \left[\frac{se^{s\varepsilon} + se^{-s\varepsilon}}{1} \right] = \frac{1}{2s} \cdot \frac{2s}{1} = 1$$

P 12.10 [a] Let $dv = \delta'(t - a) dt$, $v = \delta(t - a)$

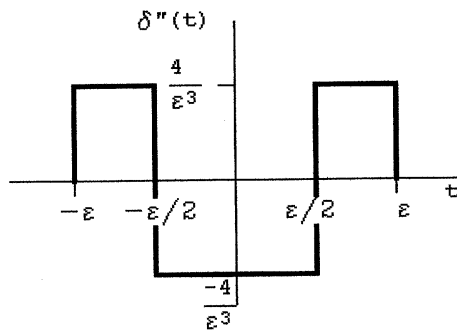
$$u = f(t), \quad du = f'(t) dt$$

Therefore

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\delta'(t - a) dt &= f(t)\delta(t - a) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t - a)f'(t) dt \\ &= 0 - f'(a) \end{aligned}$$

[b] $\mathcal{L}\{\delta'(t)\} = \int_{0^-}^{\infty} \delta'(t)e^{-st} dt = -\left[\frac{d(e^{-st})}{dt}\right]_{t=0} = -[-se^{-st}]_{t=0} = s$

P 12.11



$$F(s) = \int_{-\epsilon}^{-\epsilon/2} \frac{4}{\epsilon^3} e^{-st} dt + \int_{-\epsilon/2}^{\epsilon/2} \left(\frac{-4}{\epsilon^3}\right) e^{-st} dt + \int_{\epsilon/2}^{\epsilon} \frac{4}{\epsilon^3} e^{-st} dt$$

Therefore $F(s) = \frac{4}{s\epsilon^3} [e^{s\epsilon} - 2e^{s\epsilon/2} + 2e^{-s\epsilon/2} - e^{-s\epsilon}]$

$$\mathcal{L}\{\delta''(t)\} = \lim_{\epsilon \rightarrow 0} F(s)$$

After applying L'Hopital's rule three times, we have

$$\lim_{\epsilon \rightarrow 0} \frac{2s}{3} \left[se^{s\epsilon} - \frac{s}{4} e^{s\epsilon/2} - \frac{s}{4} e^{-s\epsilon/2} + se^{-s\epsilon} \right] = \frac{2s}{3} \left(\frac{3s}{2} \right)$$

Therefore $\mathcal{L}\{\delta''(t)\} = s^2$

P 12.12 $\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \dots$,

Therefore

$$\mathcal{L}\{\delta^n(t)\} = s^n(1) - s^{n-1}\delta(0^-) - s^{n-2}\delta'(0^-) - s^{n-3}\delta''(0^-) - \dots = s^n$$

P 12.13 [a] $\mathcal{L}\{t\} = \frac{1}{s^2}$; therefore $\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$

$$[\mathbf{b}] \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{j2}$$

Therefore

$$\begin{aligned} \mathcal{L}\{\sin \omega t\} &= \left(\frac{1}{j2}\right) \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \left(\frac{1}{j2}\right) \left(\frac{2j\omega}{s^2 + \omega^2}\right) \\ &= \frac{\omega}{s^2 + \omega^2} \end{aligned}$$

$$[\mathbf{c}] \sin(\omega t + \theta) = (\sin \omega t \cos \theta + \cos \omega t \sin \theta)$$

Therefore

$$\begin{aligned} \mathcal{L}\{\sin(\omega t + \theta)\} &= \cos \theta \mathcal{L}\{\sin \omega t\} + \sin \theta \mathcal{L}\{\cos \omega t\} \\ &= \frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2} \end{aligned}$$

$$[\mathbf{d}] \mathcal{L}\{t\} = \int_0^{\infty} t e^{-st} dt = \frac{e^{-st}}{s^2} (-st - 1) \Big|_0^{\infty} = 0 - \frac{1}{s^2} (0 - 1) = \frac{1}{s^2}$$

$$[\mathbf{e}] f(t) = \cosh t \cosh \theta + \sinh t \sinh \theta$$

From Assessment Problem 12.1(a)

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2 - 1}$$

From Assessment Problem 12.1(b)

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2 - 1}$$

$$\begin{aligned} \therefore \mathcal{L}\{\cosh(t + \theta)\} &= \cosh \theta \left[\frac{s}{s^2 - 1} \right] + \sinh \theta \left[\frac{1}{s^2 - 1} \right] \\ &= \frac{\sinh \theta + s[\cosh \theta]}{(s^2 - 1)} \end{aligned}$$

$$\begin{aligned} \text{P 12.14 } [\mathbf{a}] \mathcal{L}\{te^{-at}\} &= \int_{0^-}^{\infty} t e^{-(s+a)t} dt \\ &= \frac{e^{-(s+a)t}}{(s+a)^2} \left[-(s+a)t - 1 \right]_{0^-}^{\infty} \\ &= 0 + \frac{1}{(s+a)^2} \end{aligned}$$

$$\therefore \mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

$$[\mathbf{b}] \mathcal{L}\left\{\frac{d}{dt}(te^{-at})u(t)\right\} = \frac{s}{(s+a)^2} - 0$$

$$\mathcal{L}\left\{\frac{d}{dt}(te^{-at})u(t)\right\} = \frac{s}{(s+a)^2}$$

$$[c] \frac{d}{dt}(te^{-at}) = -ate^{-at} + e^{-at}$$

$$\mathcal{L}\{-ate^{-at} + e^{-at}\} = \frac{-a}{(s+a)^2} + \frac{1}{(s+a)} = \frac{-a}{(s+a)^2} + \frac{s+a}{(s+a)^2}$$

$$\therefore \mathcal{L}\left\{\frac{d}{dt}(te^{-at})\right\} = \frac{s}{(s+a)^2} \quad \text{CHECKS}$$

$$\begin{aligned} \text{P 12.15 [a]} \quad \mathcal{L}\{f'(t)\} &= \int_{-\varepsilon}^{\varepsilon} \frac{e^{-st}}{2\varepsilon} dt + \int_{\varepsilon}^{\infty} -ae^{-a(t-\varepsilon)}e^{-st} dt \\ &= \frac{1}{2s\varepsilon}(e^{s\varepsilon} - e^{-s\varepsilon}) - \left(\frac{a}{s+a}\right)e^{-s\varepsilon} = F(s) \end{aligned}$$

$$\lim_{\varepsilon \rightarrow 0} F(s) = 1 - \frac{a}{s+a} = \frac{s}{s+a}$$

$$[b] \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

$$\text{Therefore } \mathcal{L}\{f'(t)\} = sF(s) - f(0^-) = \frac{s}{s+a} - 0 = \frac{s}{s+a}$$

$$\text{P 12.16 } \mathcal{L}\{e^{-at}f(t)\} = \int_{0^-}^{\infty} [e^{-at}f(t)]e^{-st} dt = \int_{0^-}^{\infty} f(t)e^{-(s+a)t} dt = F(s+a)$$

$$\text{P 12.17 [a]} \quad \mathcal{L}\left\{\int_{0^-}^t e^{-ax} dx\right\} = \frac{F(s)}{s} = \frac{1}{s(s+a)}$$

$$[b] \mathcal{L}\left\{\int_{0^-}^t y dy\right\} = \frac{1}{s} \left(\frac{1}{s^2}\right) = \frac{1}{s^3}$$

$$[c] \int_{0^-}^t e^{-ax} dx = \frac{1}{a} - \frac{e^{-at}}{a}$$

$$\mathcal{L}\left\{\frac{1}{a} - \frac{e^{-at}}{a}\right\} = \frac{1}{a} \left[\frac{1}{s} - \frac{1}{s+a}\right] = \frac{1}{s(s+a)}$$

$$\int_{0^-}^t y dy = \frac{t^2}{2}; \quad \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2} \cdot \frac{2}{s^3} = \frac{1}{s^3}$$

$$\text{P 12.18 [a]} \quad \mathcal{L}\left\{\frac{d \sin \omega t}{dt} u(t)\right\} = \frac{s\omega}{s^2 + \omega^2} - \sin(0) = \frac{s\omega}{s^2 + \omega^2}$$

$$[b] \mathcal{L}\left\{\frac{d \cos \omega t}{dt} u(t)\right\} = \frac{s^2}{s^2 + \omega^2} - \cos(0) = \frac{s^2}{s^2 + \omega^2} - 1 = \frac{-\omega^2}{s^2 + \omega^2}$$

$$[c] \mathcal{L}\left\{\frac{d^3(t^2)}{dt^3} u(t)\right\} = s^3 \left(\frac{2}{s^3}\right) - s^2(0) - s(0) - 2(0) = 2$$

$$[d] \frac{d \sin \omega t}{dt} = (\cos \omega t) \cdot \omega, \quad \mathcal{L}\{\omega \cos \omega t\} = \frac{\omega s}{s^2 + \omega^2}$$

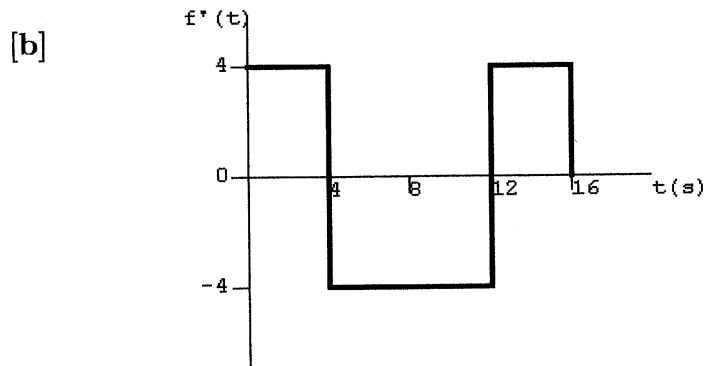
$$\frac{d \cos \omega t}{dt} = -\omega \sin \omega t$$

$$\mathcal{L}\{-\omega \sin \omega t\} = -\frac{\omega^2}{s^2 + \omega^2}$$

$$\frac{d^3(t^2)}{dt^3} = 2\delta(t); \quad \mathcal{L}\{2\delta(t)\} = 2$$

P 12.19 [a] $f(t) = 4t[u(t) - u(t - 4)]$
 $+ (32 - 4t)[u(t - 4) - u(t - 12)]$
 $+ (4t - 64)[u(t - 12) - u(t - 16)]$
 $= 4tu(t) - 8(t - 4)u(t - 4)$
 $+ 8(t - 12)u(t - 12) - 4(t - 16)u(t - 16)$

$$\therefore F(s) = \frac{4[1 - 2e^{-4s} + 2e^{-12s} - e^{-16s}]}{s^2}$$

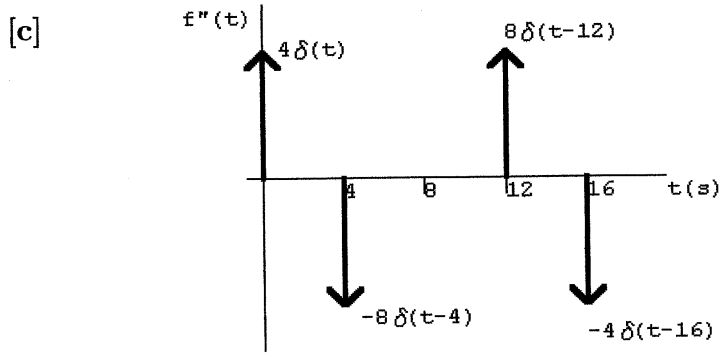


$$f'(t) = 4[u(t) - u(t - 4)] - 4[u(t - 4) - u(t - 12)]$$

$$+ 4[u(t - 12) - u(t - 16)]$$

$$= 4u(t) - 8u(t - 4) + 8u(t - 12) - 4u(t - 16)$$

$$\mathcal{L}\{f'(t)\} = \frac{4[1 - 2e^{-4s} + 2e^{-12s} - e^{-16s}]}{s}$$



$$f''(t) = 4\delta(t) - 8\delta(t-4) + 8\delta(t-12) - 4\delta(t-16)$$

$$\mathcal{L}\{f''(t)\} = 4[1 - 2e^{-4s} + 2e^{-12s} - e^{-16s}]$$

P 12.20 [a] $\int_{0^-}^t x dx = \frac{t^2}{2}$

$$\begin{aligned} \mathcal{L}\left\{\frac{t^2}{2}\right\} &= \frac{1}{2} \int_{0^-}^{\infty} t^2 e^{-st} dt \\ &= \frac{1}{2} \left[\frac{e^{-st}}{-s^3} (s^2 t^2 + 2st + 2) \right]_{0^-}^{\infty} \\ &= \frac{1}{2s^3} (2) = \frac{1}{s^3} \end{aligned}$$

$$\therefore \mathcal{L}\left\{\int_{0^-}^t x dx\right\} = \frac{1}{s^3}$$

[b] $\mathcal{L}\left\{\int_{0^-}^t x dx\right\} = \frac{\mathcal{L}\{t\}}{s} = \frac{1/s^2}{s} = \frac{1}{s^3}$

$$\therefore \mathcal{L}\left\{\int_{0^-}^t x dx\right\} = \frac{1}{s^3} \quad \text{CHECKS}$$

P 12.21 [a] $\mathcal{L}\{-20e^{-5(t-2)}u(t-2)\} = \frac{-20e^{-2s}}{(s+5)}$

[b] First rewrite $f(t)$ as

$$\begin{aligned} f(t) &= (8t - 8)u(t-1) + (24 - 8t - 8t + 8)u(t-2) \\ &\quad + (8t - 40 - 24 + 8t)u(t-4) - (8t - 40)u(t-5) \\ &= 8(t-1)u(t-1) - 16(t-2)u(t-2) \\ &\quad + 16(t-4)u(t-4) - 8(t-5)u(t-5) \end{aligned}$$

$$\therefore F(s) = \frac{8[e^{-s} - 2e^{-2s} + 2e^{-4s} - e^{-5s}]}{s^2}$$

P 12.22 $\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at)e^{-st} dt$

Let $u = at,$ $du = a dt,$ $u = 0^-$ when $t = 0^-$

and $u = \infty$ when $t = \infty$

Therefore $\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(u)e^{-(u/a)s} \frac{du}{a} = \frac{1}{a} F(s/a)$

P 12.23 [a] $f_1(t) = e^{-at} \sin \omega t;$ $F_1(s) = \frac{\omega}{(s+a)^2 + \omega^2}$

$$F(s) = sF_1(s) - f_1(0^-) = \frac{s\omega}{(s+a)^2 + \omega^2} - 0$$

[b] $f_1(t) = e^{-at} \cos \omega t;$ $F_1(s) = \frac{s+a}{(s+a)^2 + \omega^2}$

$$F(s) = \frac{F_1(s)}{s} = \frac{s+a}{s[(s+a)^2 + \omega^2]}$$

[c] $\frac{d}{dt}[e^{-at} \sin \omega t] = \omega e^{-at} \cos \omega t - a e^{-at} \sin \omega t$

Therefore $F(s) = \frac{\omega(s+a) - \omega a}{(s+a)^2 + \omega^2} = \frac{\omega s}{(s+a)^2 + \omega^2}$

$$\int_{0^-}^t e^{-ax} \cos \omega x dx = \frac{-a e^{-at} \cos \omega t + \omega e^{-at} \sin \omega t + a}{a^2 + \omega^2}$$

Therefore

$$\begin{aligned} F(s) &= \frac{1}{a^2 + \omega^2} \left[\frac{-a(s+a)}{(s+a)^2 + \omega^2} + \frac{\omega^2}{(s+a)^2 + \omega^2} + \frac{a}{s} \right] \\ &= \frac{s+a}{s[(s+a)^2 + \omega^2]} \end{aligned}$$

P 12.24 [a] $\frac{dF(s)}{ds} = \frac{d}{ds} \left[\int_{0^-}^{\infty} f(t)e^{-st} dt \right] = - \int_{0^-}^{\infty} t f(t)e^{-st} dt$

Therefore $\mathcal{L}\{t f(t)\} = - \frac{dF(s)}{ds}$

[b] $\frac{d^2 F(s)}{ds^2} = \int_{0^-}^{\infty} t^2 f(t)e^{-st} dt;$ $\frac{d^3 F(s)}{ds^3} = \int_{0^-}^{\infty} -t^3 f(t)e^{-st} dt$

Therefore $\frac{d^n F(s)}{ds^n} = (-1)^n \int_{0^-}^{\infty} t^n f(t)e^{-st} dt = (-1)^n \mathcal{L}\{t^n f(t)\}$

$$[c] \mathcal{L}\{t^5\} = \mathcal{L}\{t^4 t\} = (-1)^4 \frac{d^4}{ds^4} \left(\frac{1}{s^2} \right) = \frac{120}{s^6}$$

$$\mathcal{L}\{t \sin \beta t\} = (-1)^1 \frac{d}{ds} \left(\frac{\beta}{s^2 + \beta^2} \right) = \frac{2\beta s}{(s^2 + \beta^2)^2}$$

$\mathcal{L}\{te^{-t} \cosh t\}$:

From Assessment Problem 12.1(a),

$$F(s) = \mathcal{L}\{\cosh t\} = \frac{s}{s^2 - 1}$$

$$\frac{dF}{ds} = \frac{(s^2 - 1)1 - s(2s)}{(s^2 - 1)^2} = -\frac{s^2 + 1}{(s^2 - 1)^2}$$

$$\text{Therefore} \quad -\frac{dF}{ds} = \frac{s^2 + 1}{(s^2 - 1)^2}$$

Thus

$$\mathcal{L}\{t \cosh t\} = \frac{s^2 + 1}{(s^2 - 1)^2}$$

$$\mathcal{L}\{e^{-t} t \cosh t\} = \frac{(s + 1)^2 + 1}{[(s + 1)^2 - 1]^2} = \frac{s^2 + 2s + 2}{s^2(s + 2)^2}$$

$$\begin{aligned} \text{P 12.25 [a]} \quad \int_s^\infty F(u) du &= \int_s^\infty \left[\int_{0^-}^\infty f(t) e^{-ut} dt \right] du = \int_{0^-}^\infty \left[\int_s^\infty f(t) e^{-ut} du \right] dt \\ &= \int_{0^-}^\infty f(t) \int_s^\infty e^{-ut} du dt = \int_{0^-}^\infty f(t) \left[\frac{e^{-tu}}{-t} \Big|_s^\infty \right] dt \\ &= \int_{0^-}^\infty f(t) \left[\frac{-e^{-st}}{-t} \right] dt = \mathcal{L} \left\{ \frac{f(t)}{t} \right\} \end{aligned}$$

$$[b] \mathcal{L}\{t \sin \beta t\} = \frac{2\beta s}{(s^2 + \beta^2)^2}$$

$$\text{therefore} \quad \mathcal{L} \left\{ \frac{t \sin \beta t}{t} \right\} = \int_s^\infty \left[\frac{2\beta u}{(u^2 + \beta^2)^2} \right] du$$

Let $\omega = u^2 + \beta^2$, then $\omega = s^2 + \beta^2$ when $u = s$, and $\omega = \infty$ when $u = \infty$;
also $d\omega = 2u du$, thus

$$\mathcal{L} \left\{ \frac{t \sin \beta t}{t} \right\} = \beta \int_{s^2 + \beta^2}^\infty \left[\frac{d\omega}{\omega^2} \right] = \beta \left(\frac{-1}{\omega} \right) \Big|_{s^2 + \beta^2}^\infty = \frac{\beta}{s^2 + \beta^2}$$

$$\text{P 12.26 } i_g(t) = 5 \cos 10tu(t); \quad \text{so } I_g(s) = \frac{5s^2}{s^2 + 100}$$

$$\frac{1}{RC} = 40; \quad \frac{1}{LC} = 64; \quad \frac{1}{C} = 40$$

$$\text{Therefore } V = \frac{(40)(5)s^2}{(s^2 + 40s + 64)(s^2 + 100)} = \frac{200s^2}{(s^2 + 40s + 64)(s^2 + 100)}$$

$$\text{P 12.27 [a] } \frac{v_o - V_{dc}}{R} + \frac{1}{L} \int_0^t v_o dx + C \frac{dv_o}{dt} = 0$$

$$\therefore v_o + \frac{R}{L} \int_0^t v_o dx + RC \frac{dv_o}{dt} = V_{dc}$$

$$\text{[b] } V_o + \frac{RV_o}{Ls} + RCsV_o = \frac{V_{dc}}{s}$$

$$\therefore sLV_o + RV_o + RCLs^2V_o = LV_{dc}$$

$$\therefore V_o(s) = \frac{(1/RC)V_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

$$\text{[c] } i_o = \frac{1}{L} \int_0^t v_o dx$$

$$I_o(s) = \frac{V_o}{sL} = \frac{(1/RCL)V_{dc}}{s[s^2 + (1/RC)s + (1/LC)]}$$

$$\text{P 12.28 [a] } I_{dc} = \frac{1}{L} \int_0^t v_o dx + \frac{v_o}{R} + C \frac{dv_o}{dt}$$

$$\text{[b] } \frac{I_{dc}}{s} = \frac{V_o(s)}{sL} + \frac{V_o(s)}{R} + sCV_o(s)$$

$$\therefore V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

$$\text{[c] } i_o = C \frac{dv_o}{dt}$$

$$\therefore I_o(s) = sCV_o(s) = \frac{sI_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

$$\text{P 12.29 [a] } \frac{1}{L} \int_0^t v_1 d\tau + \frac{v_1 - v_2}{R} = i_g$$

and

$$C \frac{dv_2}{dt} + \frac{v_2}{R} - \frac{v_1}{R} = 0$$

$$[b] \frac{V_1}{sL} + \frac{V_1 - V_2}{R} = I_g$$

$$\frac{V_2 - V_1}{R} + sCV_2 = 0$$

or

$$(R + sL)V_1(s) - sLV_2(s) = RLsI_g(s)$$

$$-V_1(s) + (RCs + 1)V_2(s) = 0$$

Solving,

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

P 12.30 [a] For $t \geq 0^+$:

$$\frac{v_o}{R} + C \frac{dv_o}{dt} + i_o = 0$$

$$v_o = L \frac{di_o}{dt}; \quad \frac{dv_o}{dt} = L \frac{d^2i_o}{dt^2}$$

$$\therefore \frac{L di_o}{R dt} + LC \frac{d^2i_o}{dt^2} + i_o = 0$$

$$\text{or } \frac{d^2i_o}{dt^2} + \frac{1}{RC} \frac{di_o}{dt} + \frac{1}{LC} i_o = 0$$

$$[b] s^2 I_o(s) - sI_{dc} - 0 + \frac{1}{RC} [sI_o(s) - I_{dc}] + \frac{1}{LC} I_o(s) = 0$$

$$I_o(s) \left[s^2 + \frac{1}{RC} s + \frac{1}{LC} \right] = I_{dc} (s + 1/RC)$$

$$I_o(s) = \frac{I_{dc} [s + (1/RC)]}{[s^2 + (1/RC)s + (1/LC)]}$$

P 12.31 [a] For $t \geq 0^+$:

$$Ri_o + L \frac{di_o}{dt} + v_o = 0$$

$$i_o = C \frac{dv_o}{dt} \quad \frac{di_o}{dt} = C \frac{d^2v_o}{dt^2}$$

$$\therefore RC \frac{dv_o}{dt} + LC \frac{d^2v_o}{dt^2} + v_o = 0$$

or

$$\frac{d^2v_o}{dt^2} + \frac{R}{L} \frac{dv_o}{dt} + \frac{1}{LC} v_o = 0$$

$$[b] \quad s^2 V_o(s) - sV_{dc} - 0 + \frac{R}{L}[sV_o(s) - V_{dc}] + \frac{1}{LC}V_o(s) = 0$$

$$V_o(s) \left[s^2 + \frac{R}{L}s + \frac{1}{LC} \right] = V_{dc}(s + R/L)$$

$$V_o(s) = \frac{V_{dc}[s + (R/L)]}{[s^2 + (R/L)s + (1/LC)]}$$

P 12.32 [a] $300 = 60i_1 + 25 \frac{di_1}{dt} + 10 \frac{d}{dt}(i_2 - i_1) + 5 \frac{d}{dt}(i_1 - i_2) - 10 \frac{di_1}{dt}$

$$0 = 5 \frac{d}{dt}(i_2 - i_1) + 10 \frac{di_1}{dt} + 40i_2$$

Simplifying the above equations gives:

$$300 = 60i_1 + 10 \frac{di_1}{dt} + 5 \frac{di_2}{dt}$$

$$0 = 40i_2 + 5 \frac{di_1}{dt} + 5 \frac{di_2}{dt}$$

[b] $\frac{300}{s} = (10s + 60)I_1(s) + 5sI_2(s)$

$$0 = 5sI_1(s) + (5s + 40)I_2(s)$$

[c] Solving the equations in (b),

$$I_1(s) = \frac{60(s + 8)}{s(s + 4)(s + 24)}$$

$$I_2(s) = \frac{-60}{(s + 4)(s + 24)}$$

P 12.33 $V(s) = \frac{200s^2}{(s^2 + 40s + 64)(s^2 + 100)}$

$$s^2 + 40s + 64 = (s + 38.33)(s + 1.67); \quad s^2 + 100 = (s - j10)(s + j10)$$

Therefore

$$\begin{aligned} V(s) &= \frac{200s^2}{(s + 38.33)(s + 1.67)(s - j10)(s + j10)} \\ &= \frac{K_1}{s + 1.67} + \frac{K_2}{s + 38.33} + \frac{K_3}{s - j10} + \frac{K_3^*}{s + j10} \end{aligned}$$

$$K_1 = \left. \frac{200s^2}{(s + 38.33)(s^2 + 100)} \right|_{s=-1.67} = 0.15$$

$$K_2 = \frac{200s^2}{(s + 1.67)(s^2 + 100)} \Big|_{s=-38.33} = -5.11$$

$$K_3 = \frac{200s^2}{(s + 1.67)(s + 38.33)(s + j10)} \Big|_{s=j10} = 2.49 \angle -5.14^\circ$$

Therefore

$$v(t) = [4.98 \cos(10t - 5.14^\circ) + 0.15e^{-1.67t} - 5.11e^{-38.33t}]u(t) \text{ V}$$

P 12.34 [a] $\frac{1}{LC} = \frac{10^9}{(0.8)(100)} = 1250 \times 10^4$

$$\frac{1}{RC} = \frac{10^6}{(10)(100)} = 1000$$

$$V_o(s) = \frac{70,000}{(s^2 + 1000s + 1250 \times 10^4)}$$

$$s_{1,2} = -500 \pm \sqrt{25 \times 10^4 - 1250 \times 10^4} = -500 \pm j3500 \text{ rad/s}$$

$$\begin{aligned} V_o(s) &= \frac{70,000}{(s + 500 - j3500)(s + 500 + j3500)} \\ &= \frac{K}{s + 500 - j3500} + \frac{K^*}{s + 500 + j3500} \end{aligned}$$

$$K = \frac{70,000}{(j7000)} = 10 \angle -90^\circ$$

$$V_o(s) = \frac{10 \angle -90^\circ}{s + 500 - j3500} + \frac{10 \angle 90^\circ}{s + 500 + j3500}$$

$$v_o(t) = [20e^{-500t} \cos(3500t - 90^\circ)]u(t) \text{ V} = [20e^{-500t} \sin 3500t]u(t) \text{ V}$$

[b]
$$I_o(s) = \frac{87,500}{s(s + 500 - j3500)(s + 500 + j3500)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s + 500 - j3500} + \frac{K_2^*}{s + 500 + j3500}$$

$$K_1 = \frac{87,500}{1250 \times 10^4} = 7 \text{ mA}$$

$$K_2 = \frac{87,500}{(-500 + j3500)(j7000)} = 3.5 \angle 171.87^\circ \text{ mA}$$

$$i_o(t) = [7 + 7e^{-500t} \cos(3500t + 171.87^\circ)]u(t) \text{ mA}$$

$$\text{P 12.35 [a]} \quad \frac{1}{RC} = \frac{10^9}{(4 \times 10^3)(25)} = 10^4$$

$$\frac{1}{LC} = \frac{10^9}{(2.5)(25)} = 16 \times 10^6$$

$$V_o(s) = \frac{40 \times 10^6 I_{dc}}{s + 10,000s + 16 \times 10^6}$$

$$= \frac{40 \times 10^6 I_{dc}}{(s + 2000)(s + 8000)}$$

$$= \frac{120,000}{(s + 2000)(s + 8000)}$$

$$= \frac{K_1}{s + 2000} + \frac{K_2}{s + 8000}$$

$$K_1 = \frac{120,000}{6000} = 20; \quad K_2 = \frac{120,000}{-6000} = -20$$

$$V_o(s) = \frac{20}{s + 2000} - \frac{20}{s + 8000}$$

$$v_o(t) = [20e^{-2000t} - 20e^{-8000t}]u(t) \text{ V}$$

$$\text{[b]} \quad I_o(s) = \frac{3 \times 10^{-3}s}{(s + 2000)(s + 8000)}$$

$$= \frac{K_1}{s + 2000} + \frac{K_2}{s + 8000}$$

$$K_1 = \frac{-(3 \times 10^{-3})(2000)}{6000} = -10^{-3}$$

$$K_2 = \frac{(3 \times 10^{-3})(-8000)}{-6000} = 4 \times 10^{-3}$$

$$I_o(s) = \frac{-10^{-3}}{s + 2000} + \frac{4 \times 10^{-3}}{s + 8000}$$

$$i_o(t) = (4e^{-8000t} - e^{-2000t})u(t) \text{ mA}$$

$$\text{[c]} \quad i_o(0) = 4 - 1 = 3 \text{ mA}$$

Yes. The initial inductor current is zero by hypothesis, the initial resistor current is zero because the initial capacitor voltage is zero by hypothesis. Thus at $t = 0$ the source current appears in the capacitor.

$$\text{P 12.36} \quad \frac{1}{C} = 2 \times 10^6; \quad \frac{1}{LC} = 4 \times 10^6; \quad \frac{R}{L} = 5000; \quad I_g = \frac{0.015}{s}$$

$$V_2(s) = \frac{30,000}{s^2 + 5000s + 4 \times 10^6}$$

$$s_1 = -1000; \quad s_2 = -4000$$

$$\begin{aligned} V_2(s) &= \frac{30,000}{(s + 1000)(s + 4000)} \\ &= \frac{10}{s + 1000} - \frac{10}{s + 4000} \end{aligned}$$

$$v_2(t) = [10e^{-1000t} - 10e^{-4000t}]u(t) \text{ V}$$

$$\text{P 12.37} \quad \frac{1}{RC} = 10,000; \quad \frac{1}{LC} = 16 \times 10^6$$

$$I_o(s) = \frac{0.1(s + 10,000)}{s^2 + 10,000s + 16 \times 10^6}$$

$$s_1 = -2000; \quad s_2 = -8000$$

$$I_o(s) = \frac{0.1(s + 10,000)}{(s + 2000)(s + 8000)} = \frac{K_1}{s + 2000} + \frac{K_2}{s + 8000}$$

$$K_1 = \frac{0.1(8000)}{6000} = 0.133$$

$$K_2 = \frac{0.1(2000)}{-6000} = -0.033$$

$$I_o(s) = \frac{0.133}{s + 2000} - \frac{0.033}{s + 8000}$$

$$i_o(t) = [133.33e^{-2000t} - 33.33e^{-8000t}]u(t) \text{ mA}$$

$$\text{P 12.38} \quad \frac{R}{L} = 5000; \quad \frac{1}{LC} = 4 \times 10^6$$

$$V_o(s) = \frac{15(s + 5000)}{s^2 + 5000s + 4 \times 10^6}$$

$$s_{1,2} = -2500 \pm \sqrt{6.25 \times 10^6 - 4 \times 10^6}$$

$$s_1 = -1000 \text{ rad/s}; \quad s_2 = -4000 \text{ rad/s}$$

$$V_o(s) = \frac{15(s + 5000)}{(s + 1000)(s + 4000)} = \frac{K_1}{s + 1000} + \frac{K_2}{s + 4000}$$

$$K_1 = \frac{15(4000)}{3000} = 20 \text{ V}; \quad K_2 = \frac{15(1000)}{-3000} = -5 \text{ V}$$

$$V_o(s) = \frac{20}{s + 1000} - \frac{5}{s + 4000}$$

$$v_o(t) = [20e^{-1000t} - 5e^{-4000t}]u(t) \text{ V}$$

P 12.39 [a] $I_1(s) = \frac{K_1}{s} + \frac{K_2}{s + 4} + \frac{K_3}{s + 24}$

$$K_1 = \frac{(60)(8)}{(4)(24)} = 5; \quad K_2 = \frac{(60)(4)}{(-4)(20)} = -3$$

$$K_3 = \frac{(60)(-16)}{(-24)(-20)} = -2$$

$$I_1(s) = \left(\frac{5}{s} - \frac{3}{s + 4} - \frac{2}{s + 24} \right)$$

$$i_1(t) = (5 - 3e^{-4t} - 2e^{-24t})u(t) \text{ A}$$

$$I_2(s) = \frac{K_1}{s + 4} + \frac{K_2}{s + 24}$$

$$K_1 = \frac{-60}{20} = -3; \quad K_2 = \frac{-60}{-20} = 3$$

$$I_2(s) = \left(\frac{-3}{s + 4} + \frac{3}{s + 24} \right)$$

$$i_2(t) = (3e^{-24t} - 3e^{-4t})u(t) \text{ A}$$

[b] $i_1(\infty) = 5 \text{ A}; \quad i_2(\infty) = 0 \text{ A}$

[c] Yes, at $t = \infty$

$$i_1 = \frac{300}{60} = 5 \text{ A}$$

Since i_1 is a dc current at $t = \infty$ there is no voltage induced in the 10 H inductor; hence, $i_2 = 0$. Also note that $i_1(0) = 0$ and $i_2(0) = 0$. Thus our solutions satisfy the condition of no initial energy stored in the circuit.

$$\text{P 12.40 [a]} \quad F(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = \frac{18 - 66 + 54}{(1)(2)} = 3; \quad K_2 = \frac{72 - 132 + 54}{(-1)(1)} = 6$$

$$K_3 = \frac{162 - 198 + 54}{(-2)(-1)} = 9$$

$$\therefore f(t) = [3e^{-t} + 6e^{-2t} + 9e^{-3t}]u(t)$$

$$\text{[b]} \quad F(s) = \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+3} + \frac{K_4}{s+5}$$

$$K_1 = \left. \frac{8s^3 + 89s^2 + 311s + 300}{(s+2)(s+3)(s+5)} \right|_{s=0} = 10$$

$$K_2 = \left. \frac{8s^3 + 89s^2 + 311s + 300}{s(s+3)(s+5)} \right|_{s=-2} = 5$$

$$K_3 = \left. \frac{8s^3 + 89s^2 + 311s + 300}{s(s+2)(s+5)} \right|_{s=-3} = -8$$

$$K_4 = \left. \frac{8s^3 + 89s^2 + 311s + 300}{s(s+2)(s+3)} \right|_{s=-5} = 1$$

$$f(t) = [10 + 5e^{-2t} - 8e^{-3t} + e^{-5t}]u(t)$$

$$\text{[c]} \quad s_{1,2} = -6 \pm \sqrt{36 - 100} = -6 \pm j8$$

$$\begin{aligned} F(s) &= \frac{11s^2 + 172s + 700}{(s+2)(s+6-j8)(s+6+j8)} \\ &= \frac{K_1}{s+2} + \frac{K_2}{s+6-j8} + \frac{K_2^*}{s+6+j8} \end{aligned}$$

$$K_1 = \frac{44 - 344 + 700}{4 - 24 + 100} = 5$$

$$K_2 = \frac{11(-6+j8)^2 + 172(-6+j8) + 700}{(-4+j8)j16}$$

$$= 3 - j4 = 5 / \underline{-53.13^\circ}$$

$$\therefore f(t) = [5e^{-2t} + 10e^{-6t} \cos(8t - 53.13^\circ)]u(t)$$

$$[d] s_{1,2} = -7 \pm \sqrt{49 - 625} = -7 \pm j24$$

$$\begin{aligned} F(s) &= \frac{56s^2 + 112s + 5000}{s(s + 7 - j24)(s + 7 + j24)} \\ &= \frac{K_1}{s} + \frac{K_2}{s + 7 - j24} + \frac{K_2^*}{s + 7 + j24} \end{aligned}$$

$$K_1 = \frac{5000}{625} = 8$$

$$K_2 = \frac{56(-7 + j24)^2 + 112(-7 + j24) + 5000}{(-7 + j24)j48}$$

$$= 24 + j7 = 25/16.26^\circ$$

$$\therefore f(t) = [8 + 50e^{-7t} \cos(24t + 16.26^\circ)]u(t)$$

$$P 12.41 [a] F(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s + 10}$$

$$K_1 = \frac{8(s^2 - 5s + 50)}{s + 10} \Big|_{s=0} = \frac{400}{10} = 40$$

$$\begin{aligned} K_2 &= \frac{d}{ds} \left\{ \frac{8(s^2 - 5s + 50)}{s + 10} \right\} \Big|_{s=0} \\ &= \frac{8(s + 10)(2s - 5) - 8(s^2 - 5s + 50)(1)}{(s + 10)^2} \Big|_{s=0} \\ &= \frac{10(-40) - 8(50)}{100} = -8 \end{aligned}$$

$$K_3 = \frac{8(s^2 - 5s + 50)}{s^2} \Big|_{s=-10} = \frac{8(100 + 50 + 50)}{100} = 16$$

$$F(s) = \frac{40}{s^2} - \frac{8}{s} + \frac{16}{s + 10}$$

$$f(t) = [40t - 8 + 16e^{-10t}]u(t)$$

$$[b] F(s) = \frac{K_1}{s} + \frac{K_2}{(s + 2)^2} + \frac{K_3}{s + 2}$$

$$K_1 = \frac{10(4)}{4} = 10; \quad K_2 = \frac{10(12 - 8 + 4)}{-2} = -40$$

$$\begin{aligned} K_3 &= \frac{d}{ds} \left\{ \frac{10(3s^2 + 4s + 4)}{s} \right\} \Big|_{s=-2} \\ &= \frac{10[(s)(6s + 4) - (3s^2 + 4s + 4)]}{s^2} \Big|_{s=-2} = 20 \end{aligned}$$

$$F(s) = \frac{10}{s} - \frac{40}{(s+2)^2} + \frac{20}{s+2}$$

$$f(t) = [10 - 40te^{-2t} + 20e^{-2t}]u(t)$$

$$[c] \quad s_{1,2} = -2 \pm \sqrt{4-5} = -2 \pm j1$$

$$F(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s+2-j1} + \frac{K_3^*}{s+2+j1}$$

$$K_1 = \frac{50}{5} = 10$$

$$\begin{aligned} K_2 &= \frac{d}{ds} \left\{ \frac{s^3 - 6s^2 + 15s + 50}{s^2 + 4s + 5} \right\} \Big|_{s=0} \\ &= \frac{(s^2 + 4s + 5)(3s^2 - 12s + 15) - (s^3 - 6s^2 + 15s + 50)(2s + 4)}{(s^2 + 4s + 5)^2} \Big|_{s=0} \\ &= \frac{5(15) - 50(4)}{25} = -5 \end{aligned}$$

$$K_3 = \frac{s^3 - 6s^2 + 15s + 50}{s^2(s+2+j1)} \Big|_{s=-2+j1}$$

$$(-2+j1)^3 = -2+j11; \quad (-2+j1)^2 = 3-j4$$

$$\begin{aligned} K_3 &= \frac{-2+j11 - 6(3-j4) + 15(-2+j1) + 50}{(3-j4)(j2)} \\ &= 3+j4 = 5/\underline{53.13^\circ} \end{aligned}$$

$$F(s) = \frac{10}{s^2} - \frac{5}{s} + \frac{5/\underline{53.13^\circ}}{s+2-j1} + \frac{5/-\underline{53.13^\circ}}{s+2+j1}$$

$$f(t) = [10t - 5 + 10e^{-2t} \cos(t + 53.13^\circ)]u(t)$$

$$[d] \quad F(s) = \frac{K_1}{(s+2)^3} + \frac{K_2}{(s+2)^2} + \frac{K_3}{s+2}$$

$$K_1 = s^2 + 6s + 5 \Big|_{s=-2} = -3$$

$$K_2 = \frac{d}{ds} \{s^2 + 6s + 5\} \Big|_{s=-2} = 2s + 6 \Big|_{s=-2} = 2$$

$$2K_3 = \frac{d}{ds}(2s + 6) \Big|_{s=-2} = 2; \quad K_3 = 1$$

$$F(s) = \frac{-3}{(s+2)^3} + \frac{2}{(s+2)^2} + \frac{1}{s+2}$$

$$f(t) = -\frac{3t^2 e^{-2t}}{2} + 2te^{-2t} + e^{-2t} = [(2t - 1.5t^2 + 1)e^{-2t}]u(t)$$

$$[e] \quad s_{1,2} = -1 \pm \sqrt{1-5} = -1 \pm j2$$

$$F(s) = \frac{K_1}{(s+1-j2)^2} + \frac{K_1^*}{(s+1+j2)^2} + \frac{K_2}{s+1-j2} + \frac{K_2^*}{(s+1+j2)}$$

$$K_1 = \left. \frac{16s^3 + 72s^2 + 216s - 128}{(s+1+j2)^2} \right|_{s=-1+j2}$$

$$(-1+j2)^3 = 11-j2; \quad (-1+j2)^2 = -3-j4$$

$$K_1 = \frac{176 - j32 - 216 - j288 - 216 + j432 - 128}{-16}$$

$$= 24 - j7 = 25/\underline{-16.26^\circ}$$

$$K_2 = \frac{d}{ds} \left\{ \frac{16s^3 + 72s^2 + 216s - 128}{(s+1+j2)^2} \right\} \Big|_{s=-1+j2}$$

$$= \frac{(s+1+j2)^2(48s^2 + 144s + 216)}{(s+1+j2)^4} \Big|_{s=-1+j2}$$

$$- \frac{(16s^3 + 72s^2 + 216s - 128)2(s+1+j2)}{(s+1+j2)^4} \Big|_{s=-1+j2}$$

$$= \frac{(j4)^2(-144 - j192 - 144 + j288 + 216) - (-384 + j112)(j8)}{(j4)^4}$$

$$= \frac{2048 + j1536}{256} = 8 + j6 = 10/\underline{36.87^\circ}$$

$$F(s) = \frac{25/\underline{-16.26^\circ}}{(s+1-j2)^2} + \frac{25/\underline{16.26^\circ}}{(s+1+j2)^2} + \frac{10/\underline{36.87^\circ}}{s+1-j2} + \frac{10/\underline{-36.87^\circ}}{s+1+j2}$$

$$f(t) = [50te^{-t} \cos(2t - 16.26^\circ) + 20e^{-t} \cos(2t + 36.87^\circ)]u(t)$$

P 12.42 [a]

$$F(s) = \frac{10}{s^2 + 6s + 5} \frac{10s^2 + 85s + 95}{10s^2 + 60s + 50} \frac{10}{25s + 45}$$

$$F(s) = 10 + \frac{25s + 45}{s^2 + 6s + 5} = 10 + \frac{K_1}{s+1} + \frac{K_2}{s+5}$$

$$K_1 = \left. \frac{25s + 45}{s+5} \right|_{s=-1} = 5$$

$$K_2 = \left. \frac{25s + 45}{s+1} \right|_{s=-5} = 20$$

$$F(s) = 10 + \frac{5}{s+1} + \frac{20}{s+5}$$

$$f(t) = 10\delta(t) + [5e^{-t} + 20e^{-5t}]u(t)$$

$$[b] \quad F(s) = \frac{5}{s^2 + 4s + 5} \left[\frac{5s^2 + 40s + 25}{5s^2 + 20s + 25} \right] \frac{5}{20s}$$

$$F(s) = 5 + \frac{20s}{s^2 + 4s + 5} = 5 + \frac{K_1}{s+2-j} + \frac{K_1^*}{s+2+j}$$

$$K_1 = \frac{20s}{s+2+j} \Big|_{s=-2+j} = 10 + j20 = 22.36/63.43^\circ$$

$$F(s) = 5 + \frac{22.36/63.43^\circ}{s+2-j} + \frac{22.36/-63.43^\circ}{s+2+j}$$

$$f(t) = 5\delta(t) + 44.72e^{-2t} \cos(t + 63.43^\circ)u(t)$$

$$[c] \quad F(s) = \frac{s+5}{s+20} \left[\frac{s^2 + 25s + 150}{s^2 + 20s} \right] \frac{5s+150}{5s+100} \frac{50}{50}$$

$$F(s) = s+5 + \frac{50}{(s+20)} = s+5 + \frac{50}{s+20}$$

$$f(t) = \delta'(t) + 5\delta(t) + 50e^{-20t}u(t)$$

$$P \ 12.43 \ [a] \quad F(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s+1-j2} + \frac{K_3^*}{s+1+j2}$$

$$K_1 = \frac{100(s+1)}{s^2 + 2s + 5} \Big|_{s=0} = 20$$

$$K_2 = \frac{d}{ds} \left[\frac{100(s+1)}{s^2 + 2s + 5} \right] \Big|_{s=0} = \left[\frac{100}{s^2 + 2s + 5} - \frac{100(s+1)(2s+2)}{(s^2 + 2s + 5)^2} \right]_{s=0}$$

$$= 20 - 8 = 12$$

$$K_3 = \frac{100(s+1)}{s^2(s+1+j2)} \Big|_{s=-1+j2} = -6 + j8 = 10/126.87^\circ$$

$$f(t) = [20t + 12 + 20e^{-t} \cos(2t + 126.87^\circ)]u(t)$$

$$[b] F(s) = \frac{20s^2}{(s+1)^3} = \frac{K_1}{(s+1)^3} + \frac{K_2}{(s+1)^2} + \frac{K_3}{s+1}$$

$$\therefore 20s^2 = K_1 + K_2(s+1) + K_3(s+1)^2$$

$$K_1 = 20s^2 \Big|_{s=-1} = 20$$

After differentiating each side

$$40s = 0 + K_2 + 2K_3(s+1); \quad \therefore K_2 = 40s \Big|_{s=-1} = -40$$

After differentiating again

$$40 = 0 + 2K_3; \quad \therefore K_3 = 20$$

$$\therefore \frac{20s^2}{(s+1)^3} = \frac{20}{(s+1)^3} - \frac{40}{(s+1)^2} + \frac{20}{s+1}$$

Test at $s = 0$:

$$0 = 20 - 40 + 20 = 0 \quad \text{OK}$$

$$f(t) = \frac{20t^2 e^{-t}}{2!} - 40te^{-t} + 20e^{-t} = (10t^2 - 40t + 20)e^{-t}u(t)$$

$$[c] F(s) = \frac{K_1}{s} + \frac{K_2}{(s+1)^3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$$

$$K_1 = \frac{40(s+2)}{(s+1)^3} \Big|_{s=0} = 80$$

$$K_2 = \frac{40(s+2)}{s} \Big|_{s=-1} = -40$$

$$K_3 = \frac{d}{ds} \left[\frac{40(s+2)}{s} \right] = \left[\frac{40}{s} - \frac{40(s+2)}{s^2} \right]_{s=-1} = -40 - 40 = -80$$

$$K_4 = \frac{1}{2} \frac{d}{ds} \left[\frac{40}{s} - \frac{40(s+2)}{s^2} \right]$$

$$= \frac{1}{2} \left[\frac{-40}{s^2} - \frac{40}{s^2} + \frac{80(s+2)}{s^3} \right]_{s=-1} = \frac{1}{2} (-40 - 40 - 80) = -80$$

$$f(t) = [80 - 20t^2 e^{-t} - 80te^{-t} - 80e^{-t}]u(t)$$

$$[d] F(s) = \frac{5(s+2)^2}{s^4(s+1)} = \frac{K_1}{s+1} + \frac{K_2}{s^4} + \frac{K_3}{s^3} + \frac{K_4}{s^2} + \frac{K_5}{s}$$

$$K_1 = \frac{5(s+2)^2}{s^4} \Big|_{s=-1} = 5; \quad K_2 = \frac{5(s+2)^2}{s+1} \Big|_{s=0} = 20$$

$$\frac{5(s+2)^2}{s+1} = \frac{K_1 s^4}{s+1} + K_2 + K_3 s + K_4 s^2 + K_5 s^3$$

Differentiating each side gives

$$5 \left[\frac{(s+1)2(s+2) - (s+2)^2}{(s+1)^2} \right] = \frac{K_1 [4s^3(s+1) - s^4]}{(s+1)^2} \\ + 0 + K_3 + 2K_4 s + 3K_5 s^2$$

$$\frac{5s(s+2)}{(s+1)^2} = \frac{K_1 s^3(3s+4)}{(s+1)^2} + K_3 + 2K_4 s + 3K_5 s^2$$

$$K_3 = \frac{5s(s+2)}{(s+1)^2} \Big|_{s=0} = 0$$

Note that two more derivatives of the term involving K_1 will drop out at $s = 0$. Hence,

$$2K_4 = 5 \frac{d}{ds} \left[\frac{s(s+2)}{(s+1)^2} \right]_{s=0} - 6K_5 s \Big|_{s=0} \\ 2K_4 = 5 \left\{ \frac{(s+1)^2(2s+2) - s(s+2)2(s+1)}{(s+1)^4} \right\} \Big|_{s=0} \\ = 5(s+1) \frac{2(s+1)^2 - 2s(s+2)}{(s+1)^4} \Big|_{s=0} \\ = (5) \frac{2}{(s+1)^3} \Big|_{s=0} = 10$$

$$\therefore K_4 = 5$$

Now differentiate once more to get

$$6K_5 = \frac{d}{ds} \left\{ \frac{10}{(s+1)^3} \right\} \Big|_{s=0} \\ = \frac{-30(s+1)^2}{(s+1)^6} \Big|_{s=0} \\ = \frac{-30}{(s+1)^4} \Big|_{s=0} = -30$$

$$\therefore K_5 = -5$$

$$\frac{5(s+2)^2}{s^4(s+1)} = \frac{5}{s+1} + \frac{20}{s^4} + \frac{0}{s^3} + \frac{5}{s^2} - \frac{5}{s} \\ = \frac{5}{s+1} + \frac{20}{s^4} + \frac{5}{s^2} - \frac{5}{s}$$

Test at $s = -2$:

$$0 = -5 + \frac{20}{16} + \frac{5}{4} + \frac{5}{2} = 0 \quad \text{OK}$$

$$\therefore F(s) = \frac{5}{s+1} + \frac{20}{s^4} + \frac{5}{s^2} - \frac{5}{s}$$

$$\begin{aligned} f(t) &= 5e^{-t} + \frac{20t^3}{3!} + 5t - 5 \\ &= (5e^{5t} + \frac{10}{3}t^3 + 5t - 5)u(t) \end{aligned}$$

P 12.44 $f(t) = \mathcal{L}^{-1} \left\{ \frac{K}{s + \alpha - j\beta} + \frac{K^*}{s + \alpha + j\beta} \right\}$

$$\begin{aligned} &= Ke^{-\alpha t} e^{j\beta t} + K^* e^{-\alpha t} e^{-j\beta t} \\ &= |K| e^{-\alpha t} [e^{j\theta} e^{j\beta t} + e^{-j\theta} e^{-j\beta t}] \\ &= |K| e^{-\alpha t} [e^{j(\beta t + \theta)} + e^{-j(\beta t + \theta)}] \\ &= 2|K| e^{-\alpha t} \cos(\beta t + \theta) \end{aligned}$$

P 12.45 [a] $\mathcal{L}\{t^n f(t)\} = (-1)^n \left[\frac{d^n F(s)}{ds^n} \right]$

Let $f(t) = 1$, then $F(s) = \frac{1}{s}$, thus $\frac{d^n F(s)}{ds^n} = \frac{(-1)^n n!}{s^{(n+1)}}$

Therefore $\mathcal{L}\{t^n\} = (-1)^n \left[\frac{(-1)^n n!}{s^{(n+1)}} \right] = \frac{n!}{s^{(n+1)}}$

It follows that $\mathcal{L}\{t^{(r-1)}\} = \frac{(r-1)!}{s^r}$

and $\mathcal{L}\{t^{(r-1)} e^{-at}\} = \frac{(r-1)!}{(s+a)^r}$

Therefore $\frac{K}{(r-1)!} \mathcal{L}\{t^{r-1} e^{-at}\} = \frac{K}{(s+a)^r} = \mathcal{L} \left\{ \frac{K t^{r-1} e^{-at}}{(r-1)!} \right\}$

[b] $f(t) = \mathcal{L}^{-1} \left\{ \frac{K}{(s + \alpha - j\beta)^r} + \frac{K^*}{(s + \alpha + j\beta)^r} \right\}$

Therefore

$$\begin{aligned} f(t) &= \frac{Kt^{r-1}}{(r-1)!} e^{-(\alpha-j\beta)t} + \frac{K^*t^{r-1}}{(r-1)!} e^{-(\alpha+j\beta)t} \\ &= \frac{|K|t^{r-1}e^{-\alpha t}}{(r-1)!} [e^{j\theta}e^{j\beta t} + e^{-j\theta}e^{-j\beta t}] \\ &= \left[\frac{2|K|t^{r-1}e^{-\alpha t}}{(r-1)!} \right] \cos(\beta t + \theta) \end{aligned}$$

P 12.46 [a] $\lim_{s \rightarrow \infty} sV(s) = \lim_{s \rightarrow \infty} \left[\frac{200s^3}{s^4[1 + (40/s) + (64/s^2)][1 + (100/s^2)]} \right] = 0$

Therefore $v(0^+) = 0$

[b] Yes, all of the poles of V are in the left-half of the complex plane.

Therefore,

$$\lim_{s \rightarrow 0} sV(s) = \lim_{s \rightarrow 0} \left[\frac{200s^3}{(s^2 + 40s + 64)(s^2 + 100)} \right] = 0$$

Therefore $v(\infty) = 0$

P 12.47 [a] $sF(s) = \frac{18s^3 + 66s^2 + 54s}{(s+1)(s+2)(s+3)}$

$$\lim_{s \rightarrow 0} sF(s) = 0, \quad \therefore f(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sF(s) = 18, \quad \therefore f(0^+) = 18$$

[b] $sF(s) = \frac{8s^3 + 89s^2 + 311s + 300}{(s+2)(s^2 + 8s + 15)}$

$$\lim_{s \rightarrow 0} sF(s) = 10; \quad \therefore f(\infty) = 10$$

$$\lim_{s \rightarrow \infty} sF(s) = 8, \quad \therefore f(0^+) = 8$$

[c] $sF(s) = \frac{11s^3 + 172s^2 + 700s}{(s+2)(s^2 + 12s + 100)}$

$$\lim_{s \rightarrow 0} sF(s) = 0, \quad \therefore f(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sF(s) = 11, \quad \therefore f(0^+) = 11$$

[d] $sF(s) = \frac{56s^2 + 112s + 5000}{(s^2 + 14s + 625)}$

$$\lim_{s \rightarrow 0} sF(s) = \frac{5000}{625} = 8, \quad \therefore f(\infty) = 8$$

$$\lim_{s \rightarrow \infty} sF(s) = 56, \quad \therefore f(0^+) = 56$$

$$\text{P 12.48 [a]} \quad sF(s) = \frac{8(s^2 - 5s + 50)}{s(s + 10)}$$

$F(s)$ has a second-order pole at the origin so we cannot use the final value theorem.

$$\lim_{s \rightarrow \infty} sF(s) = 8, \quad \therefore f(0^+) = 8$$

$$\text{[b]} \quad sF(s) = \frac{10(3s^2 + 4s + 4)}{(s + 2)^2}$$

$$\lim_{s \rightarrow 0} sF(s) = \frac{40}{4} = 10, \quad \therefore f(\infty) = 10$$

$$\lim_{s \rightarrow \infty} sF(s) = 30, \quad \therefore f(0^+) = 30$$

$$\text{[c]} \quad sF(s) = \frac{s^3 - 6s^2 + 15s + 50}{s(s^2 + 4s + 5)}$$

$F(s)$ has a second-order pole at the origin so we cannot use the final value theorem.

$$\lim_{s \rightarrow \infty} sF(s) = 1, \quad \therefore f(0^+) = 1$$

$$\text{[d]} \quad sF(s) = \frac{s^3 + 6s^2 + 5s}{(s + 2)^3}$$

$$\lim_{s \rightarrow 0} sF(s) = 0, \quad \therefore f(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sF(s) = 1, \quad \therefore f(0^+) = 1$$

$$\text{[e]} \quad sF(s) = \frac{16s^4 + 72s^3 + 216s^2 - 128s}{(s^2 + 2s + 5)^2}$$

$$\lim_{s \rightarrow 0} sF(s) = 0, \quad \therefore f(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sF(s) = 16, \quad \therefore f(0^+) = 16$$

P 12.49 All of the $F(s)$ functions referenced in this problem are improper rational functions, and thus the corresponding $f(t)$ functions contain impulses ($\delta(t)$). Thus, neither the initial value theorem nor the final value theorem may be applied to these $F(s)$ functions!

$$\text{P 12.50} \quad sV_o(s) = \frac{sV_{dc}/RC}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s \rightarrow 0} sV_o(s) = 0, \quad \therefore v_o(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sV_o(s) = 0, \quad \therefore v_o(0^+) = 0$$

$$sI_o(s) = \frac{V_{dc}/RC}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s \rightarrow 0} sI_o(s) = \frac{V_{dc}/RLC}{1/LC} = \frac{V_{dc}}{R}, \quad \therefore i_o(\infty) = \frac{V_{dc}}{R}$$

$$\lim_{s \rightarrow \infty} sI_o(s) = 0, \quad \therefore i_o(0^+) = 0$$

P 12.51 $sV_o(s) = \frac{(I_{dc}/C)s}{s^2 + (1/RC)s + (1/LC)}$

$$\lim_{s \rightarrow 0} sV_o(s) = 0, \quad \therefore v_o(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sV_o(s) = 0, \quad \therefore v_o(0^+) = 0$$

$$sI_o(s) = \frac{s^2 I_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s \rightarrow 0} sI_o(s) = 0, \quad \therefore i_o(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sI_o(s) = I_{dc}, \quad \therefore v_o(0^+) = I_{dc}$$

P 12.52 $sI_o(s) = \frac{I_{dc}s[s + (1/RC)]}{s^2 + (1/RC)s + (1/LC)}$

$$\lim_{s \rightarrow 0} sI_o(s) = 0, \quad \therefore i_o(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sI_o(s) = I_{dc}, \quad \therefore i_o(0^+) = I_{dc}$$

P 12.53 [a] $sF(s) = \frac{100(s+1)}{s(s^2+2s+5)}$

$F(s)$ has a second-order pole at the origin, so we cannot use the final value theorem here.

$$\lim_{s \rightarrow \infty} sF(s) = 0, \quad \therefore f(0^+) = 0$$

[b] $sF(s) = \frac{20s^3}{(s+1)^3}$

$$\lim_{s \rightarrow 0} sF(s) = 0, \quad \therefore f(\infty) = 0$$

$$\lim_{s \rightarrow \infty} sF(s) = 20, \quad \therefore f(0^+) = 20$$

$$[\mathbf{c}] \quad sF(s) = \frac{40(s+2)}{(s+1)^3}$$

$$\lim_{s \rightarrow 0} sF(s) = 80, \quad \therefore f(\infty) = 80$$

$$\lim_{s \rightarrow \infty} sF(s) = 0, \quad \therefore f(0^+) = 0$$

$$[\mathbf{d}] \quad sF(s) = \frac{5s(s+2)^2}{s^4(s+1)} = \frac{5(s+2)^2}{s^3(s+1)}$$

$$\lim_{s \rightarrow \infty} sF(s) = 0, \quad \therefore f(0^+) = 0$$

The final value theorem cannot be applied here, as $F(s)$ violates that requirement that all poles lie in the left-half plane, with the exception of a single pole at the origin. This $F(s)$ has four poles at the origin!